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# Type II superconductivity and magnetic flux transport in neutron stars

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**ABSTRACT**
The transition to a type II proton superconductor which is believed to occur in a cooling neutron star is accompanied by changes in the equation of hydrostatic equilibrium and by the formation of proton vortices with quantized magnetic flux. Analysis of the electron Boltzmann equation for this system and of the proton supercurrent distribution formed at the transition leads to the derivation of a simple expression for the transport velocity of magnetic flux in the liquid interior of a neutron star. This shows that flux moves easily as a consequence of the interaction between neutron and proton superfluid vortices during intervals of spin-down or spin-up in binary systems. The differences between the present analysis and those of previous workers are reviewed and an error in the paper of Jones (1991) is corrected.

**Key words:** magnetic fields - stars:neutron - pulsars:general

## 1 INTRODUCTION

The origin and modes of evolution of neutron star magnetic fields have been topics of interest since the early papers of Pacini (1967) and Gold (1968). We refer to Bhattacharya & Srinivasan (1995) for a review of the many later papers published on these problems. The difficulty in the interpretation of observations on radio pulsars and binary-system neutron stars has been that the magnetic flux transport properties of both the solid crust and the liquid interior are not well known. Empirical deduction of these properties is not feasible and *a priori* theoretical input is required. A series of papers on the structure of the solid phase in the neutron-drip region of the inner crust have shown that it is amorphous, heterogeneous in nuclear charge $Z$, with a high temperature-independent resistivity (Jones 2004). But there remain uncertainties about the movement of flux in the liquid interior, the problem with which this paper is concerned.

The outer region of the liquid core is believed to consist of a normal system of ultra-relativistic electrons, a ${}^1S_0$ or ${}^3P_2$ neutron superfluid, and a ${}^1S_0$ proton superfluid (see Baym & Pethick 1979). Negative $\mu$-mesons, essentially non-relativistic, are also present if the electron chemical potential exceeds their rest energy. Our assumption, following the early paper of Baym, Pethick & Pines (1969), is that the protons form a type II superconductor. Type I superconductivity will be present in any density region for which $\kappa = \lambda/\xi < 1/\sqrt{2}$, where $\lambda$ is the proton penetration depth and $\xi$ the coherence length, but this condition can be satisfied only for very small proton energy gaps. During the early stages of neutron-star cooling, the proton phase transition, from normal to superconducting Fermi liquid, is accompanied by the formation of a mixed state. On microscopic scales, magnetic flux becomes quantized. The quantum of magnetic flux is $\phi_0 = hc/2e = 2.07 \times 10^{-7}$ G cm$^2$, and it is confined to the core of a proton vortex. The highest-density regions of the inner crust may consist of low-dimensional structures (see Pethick & Ravenhall 1995) in place of spherical nuclei. In this case, the boundary of the proton superconductor is not well-defined but we shall nevertheless assume the existence of a spherical surface within which macroscopic supercurrents can flow. The movement of flux across this surface depends, of course, on the $B(H)$ characteristic of the superconductor and the value of its lower critical field $H_{c1}$ (see, for example, Fetter & Hohenberg 1969).

Apart from the effects of ohmic diffusion and Hall drift of the field, we may assume that an approximation to static hydrodynamic equilibrium exists at times before the superconducting phase transition. But the transition changes this equilibrium because the components of the proton superconductor stress tensor are larger than those of the normal-system Maxwell tensor by factors of the order of $H_{c1}/B$, where $B$ is the mean magnetic flux density of the superconductor mixed state (Jones 1975, Easson & Pethick 1977). The divergence of the superconductor stress tensor is a volume force and, with the different transport properties of the superconducting system, produces a proton-vortex drift velocity. The force has been referred to as a buoyancy force and was first estimated by Muslimov & Tsygan (1985). They

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obtained a drift velocity from its steady-state equilibrium with a viscous force derived from the magnetic scattering of electrons by an isolated vortex. This specific problem was later re-examined by Harvey, Ruderman & Shaham (1986). Further work by Jones (1987, 1991) and by Harrison (1991) showed that the magnetic viscous force acting on an isolated vortex moving relative to the electrons is not relevant to the problem of obtaining a drift velocity and that it is necessary to consider the interaction of the whole system of vortices which is arranged as a two-dimensional lattice. The different ways in which magnetic flux might be expelled, primarily from normal Fermi systems, were analyzed by Goldreich & Reisenegger (1992) who emphasized the importance of the stable stratification of neutron-star matter and introduced the concept of ambipolar diffusion, that is, drift of the charged components of a plasma relative to the neutral such as may occur in the interstellar medium (see, for example, Spitzer 1968). The severe constraint imposed on ambipolar diffusion by stable stratification was the motivation for the work of Ruderman, Zhu & Chen (1998).

There are several reasons for writing a further paper on this problem. The papers cited above, in so far as they consider type II superconductivity, are contradictory. There is also an error in the paper by Jones (1991). Given the quantity of observational data on radio pulsars and on binary systems, there is a possibility that a clear solution to the problem might reveal interesting information about the neutron star interior. The idea that proton vortices move easily as a result of interaction with neutron vortices during intervals of spin-down or spin-up is widely assumed in studies of X-ray binary systems (see Bhattacharya & Srinivasan 1995) and unambiguous theoretical confirmation of its validity, or otherwise, is desirable.

The approach to the problem made in the present paper uses the fact that, at the temperature considered here, all superfluid quasiparticles except those localized in vortex cores have negligibly small number densities. Thus it is possible to write down a Boltzmann equation for the electrons (muons) interacting only with muons (electrons) and with the proton vortices and obtain a steady-state solution (Section 2). The screening of the lepton current in the superconductor and the proton force-balance give further equations connecting drift velocities and chemical potential gradients for leptons and protons (Section 3). A simple and unambiguous result is obtained for the drift velocity of the proton vortices (equation 22) and hence for the expulsion of flux from the interior of the neutron star. It is the principal result of this paper, and is shown to be unaffected by possible lepton interactions with other degrees of freedom. In Section 4, we attempt to analyze the differences between this paper and the work of each of the sets of authors cited above. The scope of the paper is purely technical and it does not consider observational evidence relevant to magnetic flux evolution. It excludes the possibility of type I superconductivity induced by interaction between neutron and proton superfluids (Buckley, Metlitski & Zhitnitsky 2004), also systems of several distinct superconductors which may be present at very high matter densities in the inner liquid core. Unless otherwise stated, all quantities are defined in a coordinate system corotating with the solid crust of the star and having angular velocity $\mathbf{\Omega}$.

## 2 THE LEPTON BOLTZMANN EQUATION

The reference system we shall use within the chosen coordinates is the steady state with zero vortex drift velocity, $v_L = 0$. A local definition of electron chemical potential is used, $\mu_e = \epsilon_{Fe}$, where $\epsilon_{Fe}$ is the Fermi energy, including rest energy. For present purposes, this is more convenient than the global definition, $\mu_e = (\epsilon_{Fe} - e\phi)\sqrt{g_{00}}$, in terms of the time-like component of the metric tensor and the gravitationally-induced electric potential $\phi$. This latter definition was favoured by Harrison (1991); its equilibrium value $m_e c^2\sqrt{g_{00s}}$, defined at the stellar surface, is constant throughout the star. In the liquid core, $\epsilon_{Fe}$ can exceed the muon rest energy so that a muon chemical potential $\mu_\mu$ is also defined. At these energies, lepton transport can be considered under the assumption of non-quantizing fields (see Potekhin 1999). The classical leptonic orbits in a plane perpendicular to the proton vortices are irregular polygons whose vertices represent scattering by the microscopic magnetic flux density $\tilde{\mathbf{B}}$ localized within the vortex cores. To be specific, we consider first the electron component. Its orbit size can be specified by the orbit radius in the spatially-averaged magnetic flux density $\mathbf{B} = \langle\tilde{\mathbf{B}}\rangle$. This is $r_B = \epsilon_{Fe}/eB = 3.3 \times 10^{-7} B_{12}^{-1}$ cm, for $\epsilon_{Fe} = 100$ MeV, where $B_{12}$ is the magnetic flux density in units of $10^{12}$ G. It is several orders of magnitude larger than the intervortex spacing in the triangular lattice, $d = 4.9 \times 10^{-10} B_{12}^{-1/2}$ cm. For such a system, it is possible to define, within a small element of phase space, a spatially-dependent electron Fermi distribution function which satisfies a Boltzmann equation (see, for example, Pines & Nozières 1966). In the steady state, with $v_L = 0$ and isotropic distribution function $n_k^0$, this can be expressed as

$$\mathbf{v_k} \cdot \nabla n_k^0 - e\left(\boldsymbol{\mathcal{E}}_0 + \frac{1}{c}\mathbf{v_k} \times \mathbf{B}\right) \cdot \nabla_{\mathbf{k}} n_k^0 = -\Gamma_{\mathbf{k}}^0 + e\left(\frac{1}{c}\mathbf{v_k} \times \left(\tilde{\mathbf{B}} - \mathbf{B}\right)\right) \cdot \nabla_{\mathbf{k}} n_k^0, \quad (1)$$

with electron velocity $\mathbf{v_k}$ for momentum $\mathbf{k}$. The conservative field $\boldsymbol{\mathcal{E}}_0$ cancels the $\nabla n_k^0$ term in the equation. Motion of the vortex lattice produces an induction field $\tilde{\mathbf{E}} = -(1/c)\mathbf{v}_L \times \tilde{\mathbf{B}}$ and changes the electron distribution function to

$$n_{\mathbf{k}} = n_k^0 + \beta n_k^0(1 - n_k^0)\delta\mu_e + \delta n_{\mathbf{k}}, \quad (2)$$

in which the first incremental term is isotropic, with $\beta^{-1} = k_B T$, and represents the effect of a chemical potential change $\delta\mu_e = \delta\epsilon_{Fe} - e\delta\phi$. The second term is anisotropic. The modified Boltzmann equation is then,

$$\mathbf{v_k} \cdot \nabla n_{\mathbf{k}} - e\left(\boldsymbol{\mathcal{E}}_0 + \mathbf{E} + \frac{1}{c}\mathbf{v_k} \times \mathbf{B}\right) \cdot \nabla_{\mathbf{k}} n_{\mathbf{k}} = -\Gamma_{\mathbf{k}} + e\left(\tilde{\mathbf{E}} - \mathbf{E} + \frac{1}{c}\mathbf{v_k} \times \left(\tilde{\mathbf{B}} - \mathbf{B}\right)\right) \cdot \nabla_{\mathbf{k}} n_{\mathbf{k}} \quad (3)$$

Both equations have been expressed in terms of the spatially-averaged fields. The reason for this is that the difference between the irregular polygons and the circular orbits of the spatially-averaged field can be thought of as the consequence of a scattering process. Therefore, the difference terms producing it are placed on the right-hand side of each equation with the collision integrals $\Gamma_{\mathbf{k}}^0$ and $\Gamma_{\mathbf{k}}$. These are derived from processes other than magnetic scattering and include electromagnetic scattering by muons (if present) and by vortex-core quasiparticles. But these Boltzmann equa-

tions are for a distribution function $n_{\mathbf{k}}(\mathbf{r})$ defined within a small element of phase space. Spatial integration over that small element leaves the terms in them unchanged with the exception of the difference terms, whose integrals clearly converge to zero as the linear dimension of the element becomes large compared with the intervortex spacing $d$. This serves to establish the intuitively obvious result (Harrison 1991, Jones 1991) that, given the condition $r_B \gg d$, the microscopic fields $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ can be replaced without error by the spatial averages $\mathbf{E}$ and $\mathbf{B}$.

With neglect of some terms of the second order of smallness, the incremental change in distribution function $\delta n_{\mathbf{k}}$ satisfies

$$-e\delta\boldsymbol{\mathcal{E}}\cdot\nabla_{\mathbf{k}}n_k^0 - \frac{e}{c}\left(\mathbf{v}_{\mathbf{k}}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{k}}\delta n_{\mathbf{k}} = \Gamma_{\mathbf{k}}^0 - \Gamma_{\mathbf{k}}, \tag{4}$$

where,

$$\delta\boldsymbol{\mathcal{E}} = \mathbf{E} + \frac{1}{e}\nabla\delta\mu_e. \tag{5}$$

In order to write down an explicit form for the collision integral difference, it is necessary to specify each set of degrees of freedom with which the electrons interact. At temperatures well below the superconducting transition, superfluid quasiparticle densities are negligible except for those localized in vortex cores. The rest frame for these excitations is therefore that of the vortices, with velocity $\mathbf{v}_L$ in the reference frame corotating with the solid crust in which our quantities are defined. Thus interaction with them causes $\delta n_{\mathbf{k}}$ to relax to isotropy in the rest frame of the vortex lattice with relaxation time $\tau_v^e$. There is also interaction with muons (if present) and possibly with other excitations, such as zero sound phonons or other collective modes, or with superfluid continuum quasiparticles whose density will be significant near the transition temperature. In order to include them, we assume for the muons a natural rest frame with velocity $\mathbf{v}_\mu$, in which their thermal-equilibrium distribution function is isotropic, and a separate relaxation time $\tau_\mu^e$. For the unspecified excitations, we assume a rest-frame velocity $\mathbf{v}_u$ and relaxation time $\tau_u^e$, with both quantities treated as unknowns. Thus the collision integral difference is

$$\begin{aligned}\Gamma_{\mathbf{k}}^0 - \Gamma_{\mathbf{k}} \quad = \quad & -\frac{1}{\tau_v^e}\left(\delta n_{\mathbf{k}} - \beta n_k^0(1-n_k^0)\mathbf{k}\cdot\mathbf{v}_L\right)\\ & -\frac{1}{\tau_\mu^e}\left(\delta n_{\mathbf{k}} - \beta n_k^0(1-n_k^0)\mathbf{k}\cdot\mathbf{v}_\mu\right)\\ & -\frac{1}{\tau_u^e}\left(\delta n_{\mathbf{k}} - \beta n_k^0(1-n_k^0)\mathbf{k}\cdot\mathbf{v}_u\right).\end{aligned} \tag{6}$$

Equations (4) and (6) are satisfied by,

$$\delta n_{\mathbf{k}} = e\tau_v^e\frac{\partial n_k^0}{\partial\epsilon_k}\mathbf{v}_{\mathbf{k}}\cdot\mathbf{A}, \tag{7}$$

where $\mathbf{A}$ is related to the electron current density,

$$\mathbf{J}^e = -N_e e\mathbf{v}_e = \frac{-2e}{(2\pi)^3}\int d^3k\,\mathbf{v}_{\mathbf{k}}n_{\mathbf{k}} = \sigma_v^e\mathbf{A}, \tag{8}$$

and $N_e$ is the electron number density. Equations (8) define the electron drift velocity $\mathbf{v}_e$. The resistivities $\left(\sigma_j^e\right)^{-1}$ are defined by

$$\sigma_j^e = \frac{N_e e^2\tau_j^e c^2}{\epsilon_{Fe}}, \tag{9}$$

where the subscript denotes the system with which the electrons interact. The relation between the drift velocities $\mathbf{v}_{e,\mu,u}$ and $\mathbf{v}_L$ derived from equations (4) and (6) can be expressed in the form,

$$\begin{aligned}\frac{1}{e}\nabla\delta\mu_e - \frac{N_e e}{\sigma_\mu^e}\left(\mathbf{v}_\mu - \mathbf{v}_e\right) - \frac{N_e e}{\sigma_u^e}\left(\mathbf{v}_u - \mathbf{v}_e\right) = \\ \frac{N_e e}{\sigma_v^e}\left(\mathbf{v}_L - \mathbf{v}_e\right) + \frac{1}{c}\left(\mathbf{v}_L - \mathbf{v}_e\right)\times\mathbf{B}.\end{aligned} \tag{10}$$

This unremarkable result is no more than a force-balance equation relating the force on the electrons from the vortex lattice (right-hand side) with the forces exerted by the electrons on the chemical potential gradient, on the muons, and on the unspecified system of excitations whose rest-frame has velocity $\mathbf{v}_u$. A similar equation exists for the muons,

$$\begin{aligned}\frac{1}{e}\nabla\delta\mu_\mu - \frac{N_\mu e}{\sigma_e^\mu}\left(\mathbf{v}_e - \mathbf{v}_\mu\right) - \frac{N_\mu e}{\sigma_u^\mu}\left(\mathbf{v}_u - \mathbf{v}_\mu\right) = \\ \frac{N_\mu e}{\sigma_v^\mu}\left(\mathbf{v}_L - \mathbf{v}_\mu\right) + \frac{1}{c}\left(\mathbf{v}_L - \mathbf{v}_\mu\right)\times\mathbf{B}.\end{aligned} \tag{11}$$

The rate at which the lattice, the muons and the unspecified excitations do work on unit volume of the electrons less the rate at which the electron system does work in moving on the chemical potential gradient is,

$$\begin{aligned}N_e e\mathbf{v}_L\cdot\left(\frac{N_e e}{\sigma_v^e}\left(\mathbf{v}_L - \mathbf{v}_e\right) + \frac{1}{c}\left(\mathbf{v}_L - \mathbf{v}_e\right)\times\mathbf{B}\right)\\ -N_e\mathbf{v}_e\cdot\nabla\delta\mu_e + \frac{(N_e e)^2}{\sigma_\mu^e}\left(\mathbf{v}_\mu - \mathbf{v}_e\right)\cdot\mathbf{v}_\mu\\ +\frac{(N_e e)^2}{\sigma_u^e}\left(\mathbf{v}_u - \mathbf{v}_e\right)\cdot\mathbf{v}_u = \\ \frac{(N_e e)^2}{\sigma_v^e}\left(\mathbf{v}_L - \mathbf{v}_e\right)^2 + \frac{(N_e e)^2}{\sigma_\mu^e}\left(\mathbf{v}_\mu - \mathbf{v}_e\right)^2\\ +\frac{(N_e e)^2}{\sigma_u^e}\left(\mathbf{v}_u - \mathbf{v}_e\right)^2,\end{aligned} \tag{12}$$

which is the rate of energy dissipation per unit volume attributable to $\mathbf{v}_e$. A similar relation exists for the muons.

The analogous force-balance equation for the proton superfluid continuum can be written down immediately because the continuum quasiparticle number density is negligibly small. It relates the chemical potential gradient with the Magnus force per unit volume,

$$N_p\nabla\delta\mu_p = \frac{N_p e}{c}\left(\mathbf{v}_{p0} - \mathbf{v}_L\right)\times\mathbf{B}, \tag{13}$$

in which $\mathbf{v}_{p0}$ is the proton drift velocity. Equations (10), (11) and (13) are the principal results of this Section but contain a total of 7 variables excluding $\mathbf{v}_u$. The further equations required can be obtained by examining the effect of the normal to superconducting transition on the various current densities in the crust and liquid interior.

## 3 STEADY-STATE EQUILIBRIUM OF THE MAGNETIC FLUX DISTRIBUTION

Before the transition to superconductivity, the magnetic flux and lepton current density are slowly-varying functions of position satisfying Ampère's theorem. If the transition is to a type II superconductor, it is accompanied on a microscopic scale by the formation of proton vortices. These have quantized magnetic flux $\phi_0$, locally collinear with their axes, supported by a circulating supercurrent distribution whose

density decreases exponentially with radius on the scale of the penetration depth $\lambda$. If, on macroscopic scales, the vortex number density $N_v$ is such that the spatially-averaged magnetic flux density is unchanged, so that $N_v\boldsymbol{\phi}_0 = \mathbf{B}$, the spatial average of the individual vortex supercurrent distributions $\tilde{\mathbf{J}}^{p\alpha}$ must satisfy the relation,

$$\left\langle \sum_{\alpha} \tilde{\mathbf{J}}^{p\alpha} \right\rangle = \mathbf{J}^e + \mathbf{J}^\mu. \tag{14}$$

The pre-existing lepton current density must be screened out by a supercurrent density $\mathbf{J}^{p0}$ which, in the body of the superconductor, has significant variation only over lengths many orders of magnitude larger than $\lambda$,

$$\mathbf{J}^{p0} + \mathbf{J}^e + \mathbf{J}^\mu = 0. \tag{15}$$

This ensures that Ampère's theorem is satisfied at all points in space (see Jones 1991) and, as a consequence of superconductivity, is analogous with the formation of surface current sheets in the Meissner effect. An additional condition expresses the fact that protons do not cross the superconductor boundary,

$$\left( \mathbf{J}^{p0} + \sum_{\alpha} \tilde{\mathbf{J}}^{p\alpha} \right)_{\perp} = 0. \tag{16}$$

The effect of equation (16) is that the supercurrent distribution $\mathbf{J}^{p0}$ defined by equation (15) must be associated with a return current sheet at the boundary separating the superconductor from the normal solid. This current sheet, which has negligible kinetic energy, also maintains the mixed state of the superconductor by excluding the continuous magnetic flux density of the normal system.

The neutrality condition $N_e + N_\mu = N_p$ can be regarded as exact and equation (15) gives the condition $N_e\mathbf{v}_e + N_\mu\mathbf{v}_\mu = N_p\mathbf{v}_{p0}$ which is satisfied at all points in the superconductor by the lepton and proton drift velocities. The final equation is for steady-state equilibrium of the vortex lattice under what is referred to as the buoyancy force. The argument for the existence of this quantity is as follows. The normal-state hydrostatic equilibrium before the transition is given by the equation,

$$\rho g_i - \frac{\partial P^0}{\partial x_i} + \frac{1}{c}\left( (\mathbf{J}^e + \mathbf{J}^\mu) \times \mathbf{B} \right) = 0, \tag{17}$$

in terms of the matter density $\rho$, the gravitational acceleration $\mathbf{g}$, and the zero-field pressure $P^0$. The third term is the divergence of the Maxwell stress tensor. In hydrostatic equilibrium, its curl is almost exactly a non-radial vector. (This condition does not apply in the crust owing to the presence there of a further term, derived from the solid stress tensor, which allows more general components in the divergence of the Maxwell tensor and the resultant phenomenon of Hall drift. In the liquid, $\nabla \times ((\mathbf{J}^e + \mathbf{J}^\mu) \times \mathbf{B})$ can produce Hall drift only of non-radial components of $\mathbf{B}$. Here, for brevity, components in the electromagnetic current density other than $\mathbf{J}^{e,\mu}$ have been neglected.) At the transition, the magnetic part of the superconductor stress tensor replaces the Maxwell tensor. Following Easson & Pethick (1977), we express this in the form,

$$T^S_{ij} = -P^S\delta_{ij} + \frac{1}{4\pi}H_iB_j, \tag{18}$$

which is a symmetric tensor because $\mathbf{B}$ and $\mathbf{H}$ are locally parallel vectors. Both isotropic and anisotropic components are larger than the corresponding Maxwell components by factors of the order of $H_{c1}/B \gg 1$. The new hydrostatic equilibrium is given by

$$\rho g_i - \frac{\partial P^0}{\partial x_i} + \frac{\partial T^S_{ij}}{\partial x_j} + f_{Vi} = 0, \tag{19}$$

where $\mathbf{f}_V$ is the force per unit volume arising from interaction between neutron and proton vortices (Sauls 1989), an additional effect which is not included in equation (18). (Strictly, the existence of $\mathbf{f}_V \neq 0$ depends on rotation of the neutron superfluid with angular velocity $\boldsymbol{\Omega}_n \neq \boldsymbol{\Omega}$.) The derivative of the isotropic component of the stress tensor has been referred to as a buoyancy force, but the implicit neglect of the anisotropic component means that this description is not necessarily apt. Easson & Pethick obtained an expression (equation 17 of their paper) for the isotropic component of the stress tensor valid for any $\kappa > 1/\sqrt{2}$ in the limit $B \ll H_{c1}$. But they note that it can be estimated reliably only in the extreme type II limit in which $\kappa \gg 1$. Thus for more general values of $\kappa$, it is not obvious that the term leads to a buoyancy force, rather than the reverse. In the limit $B \ll H_{c1}$, the magnetic field $H \approx H_{c1}$ and we can assume that the spatial derivatives of its magnitude (though not direction) are much smaller than those of $B$. With neglect of these terms, and in the further limit $\kappa \gg 1$, the divergence of the stress tensor given by equation (18) is,

$$\frac{\partial T^S_{ij}}{\partial x_j} = \frac{1}{4\pi}\left( (\nabla \times \mathbf{B}) \times \mathbf{H}_{c1} \right)_i + \frac{1}{4\pi}\left( \nabla \cdot \mathbf{H}_{c1} \right) B_i, \tag{20}$$

in which $\mathbf{H}_{c1}$ has magnitude $H_{c1}$ and is everywhere parallel with $\mathbf{B}$. This replaces the third term in equation (19). The sum of the third and fourth terms in equation (19) is the total magnetic flux-dependent force per unit volume (defined as $\mathbf{f}_B$) and acts on the vortex lattice. Its direction is not necessarily the radial direction expected of a buoyancy force and it is analogous with the third term in equation (17).

The lattice drift velocity $\mathbf{v}_L$ is then defined by equating $\mathbf{f}_B$ with the sum of the force on the electron and muon systems and the Magnus force on the superfluid continuum,

$$\begin{aligned} \mathbf{f}_B - \frac{(N_e e)^2}{\sigma^e_v}(\mathbf{v}_L - \mathbf{v}_e) - \frac{N_e e}{c}(\mathbf{v}_L - \mathbf{v}_e) \times \mathbf{B} \\ - \frac{(N_\mu e)^2}{\sigma^\mu_v}(\mathbf{v}_L - \mathbf{v}_\mu) - \frac{N_\mu e}{c}(\mathbf{v}_L - \mathbf{v}_\mu) \times \mathbf{B} \\ - \frac{N_p e}{c}(\mathbf{v}_{p0} - \mathbf{v}_L) \times \mathbf{B} = 0, \end{aligned} \tag{21}$$

which gives an extremely simple final result,

$$\mathbf{v}_L = \frac{\sigma^\mu_v N_e^2 \mathbf{v}_e + \sigma^e_v N_\mu^2 \mathbf{v}_\mu}{\sigma^\mu_v N_e^2 + \sigma^e_v N_\mu^2} + \frac{\tilde{\sigma}}{(N_e + N_\mu)^2 e^2}\mathbf{f}_B, \tag{22}$$

where,

$$\tilde{\sigma} = \frac{(N_e + N_\mu)^2 \sigma^e_v \sigma^\mu_v}{\sigma^\mu_v N_e^2 + \sigma^e_v N_\mu^2}. \tag{23}$$

It is interesting that this result is quite independent of the existence of the resistivities derived from lepton interaction with other degrees of freedom whose natural rest frame is not that of the vortex lattice. Equations (10), (11), (13), (15) and (21) give $\mathbf{v}_L$ and a complete description of the system in terms of the independent variables $\mathbf{v}_e$, $\mathbf{v}_\mu$ and $\mathbf{f}_B$,

which all depend on the properties of the normal-state $\mathbf{B}$-distribution that existed before the superconducting transition. Scattering transition rates for $e-p$ and $\mu-p$ at the Fermi surfaces, though of the same order of magnitude, are not identical so that we anticipate $\mathbf{v}_e \neq \mathbf{v}_\mu$ and retain both as independent variables in equation (22). Of the remaining variables $\mathbf{v}_{p0}$ and $\nabla\delta\mu_{e,\mu,p}$, the increments in chemical potential, $\delta\mu_e(\delta N_e, \delta\phi)$, $\delta\mu_\mu(\delta N_\mu, \delta\phi)$ and $\delta\mu_p(\delta N_p, -\delta\phi)$, are functions of just 3 further variables $\delta N_e$, $\delta N_\mu$ and $\delta\phi$. The values of $\tilde{\sigma}$ that are immediately relevant are those at temperatures below, but within an order of magnitude of the superconducting transition temperature. We refer to Jones (1991) for the order of magnitude of $\sigma^e_v$ and for a brief discussion of the factors determining its temperature dependence. But it is possible to assert that as the star cools, it increases at least as rapidly as $T^{-2}$, where $T$ is the temperature. Thus at a typical transition temperature of $3\times10^9$ K, the order of magnitude is $\tilde{\sigma} \approx 10^{29}B_{12}^{-1}$ s$^{-1}$. A force with order of magnitude $f_B \approx 10^{20}B_{12}$ dyne cm$^{-3}$ then leads, from equation (22), to a velocity $v_L \approx 4\times10^{-7}$ cm s$^{-1}$. It is evident that $\tilde{\sigma}$, and therefore $v_L$ both increase rapidly as the star cools. Examination of the orders of magnitude present in equations (5), (10) and (11) shows that the chemical potential gradients can become quite large. But they are almost exactly cancelled by the induction field. Our conclusion is that, for any plausible value of $f_B$, the post-transition movement of magnetic flux is fast in comparison with, for example, radio pulsar spin-down or binary-system evolutionary time-scales.

## 4 CONCLUSIONS

Before proceeding to a comparison of the results obtained here with those of earlier papers, it is worth considering two questions about flux movement. Konenkov & Geppert (2001) observed that proton vortices, given a buoyancy force, always move toward the crust by sliding (with some dissipation) along the rectilinear neutron vortices of the rotating neutron superfluid. (The only exceptional case is that in which proton vortices, within a small element of solid angle, are approximately parallel with the neutron vortices.) This interesting possibility clearly depends on $\mathbf{f}_B$ having a component, parallel with the neutron vortices and the spin vector $\mathbf{\Omega}_n$ of the neutron superfluid, with the appropriate sign. But we have seen, both from the discussion given by Easson & Pethick and by examination of equation (20), that this sign must depend on the form of the flux distribution prior to the superconducting transition. For example, the force terms in equation (20) vanish in the case of a uniform $\mathbf{B}$, giving $\mathbf{f}_B = \mathbf{f}_V$, which is almost exactly perpendicular to $\mathbf{\Omega}_n$. The second question concerns whether or not proton vortices can be moved inward or outward, relative to the rotation axis of the star, by interaction with neutron vortices during intervals of spin-up or spin-down. The neutron vortices at a distance $r_\perp$ from the rotation axis move with a radial velocity component $v_\perp = -r_\perp\dot{\Omega}_n/2\Omega_n$, where $\mathbf{\Omega}_n$ is the neutron superfluid angular velocity at $r_\perp$. From equation (22), $\mathbf{v}_L$ can have a component of this size provided,

$$\tilde{f}_V\tilde{\sigma} > \frac{\pi r_\perp \mid \dot{\Omega} \mid \hbar e^2(N_e+N_\mu)^2}{4m_p\Omega^2}, \tag{24}$$

where $m_p$ is the proton mass and $\tilde{f}_V$ is the maximum force that unit length of neutron vortex can exert on a lattice of proton vortices without intersection. (Here, we assume $\mathbf{\Omega}_n = \mathbf{\Omega}$, the angular velocity of the star.) It is easy to confirm that this is clearly satisfied in the case of the observed radio pulsars by evaluation for the Crab, which gives $\tilde{f}_V\tilde{\sigma} > 8\times10^{44}$ dyne cm$^{-1}$ s$^{-1}$. A value $\tilde{f}_V \sim 10^{14}$ dyne cm$^{-1}$ is a conservative assumption (see Jones 1991) which, with the estimated $\tilde{\sigma} > 10^{32}B_{12}^{-1}$ s$^{-1}$ for $N_\mu \ll N_e$, shows that the condition is satisfied. However, much higher spin-down rates exist during the propeller phase of binary systems. The model of Urpin, Geppert & Konenkov (1997) gives

$$\dot{\Omega} = -(GM)^{3/7}R^{6/7}B^{2/7}\dot{M}^{6/7}I^{-1}, \tag{25}$$

where $M$, $R$ and $I$ are, respectively, the neutron star mass, radius and moment of inertia. For typical values ($M = 1.4M_\odot$, $R = 1.2\times10^6$ cm, $I = 10^{45}$ g cm$^2$ and an accretion rate on to the Alfvén surface of $\dot{M} = 10^{-10}M_\odot$ yr$^{-1}$) the inequality (24) gives,

$$\tilde{f}_V\tilde{\sigma} > 3\times10^{46}B_{12}^{2/7}\Omega^{-2}, \tag{26}$$

that is, $\tilde{f}_V\tilde{\sigma} > 8\times10^{48}$ dyne cm$^{-1}$ s$^{-1}$ for the rotation periods of $10^2$ s that are observed at the end of the spin-down phase of binary evolution (see the review of Verbunt & van den Heuvel 1995). Satisfaction of this condition is much more problematic.

Previous papers concerned with the flux transport velocity have reached differing conclusions. In their seminal paper on type II superconductivity in neutron stars, Baym, Pethick & Pines assumed that the viscous force on unit length of moving proton vortex would arise from a dissipation rate of the order of

$$\pi\xi^2\sigma\tilde{E}^2 \sim \phi_0 H_{c2}\sigma\frac{v_L^2}{c^2}, \tag{27}$$

analogous with the theory of Bardeen & Stephen (1965). In equilibrium with the force per unit length $f_B\phi_0/B$ derived from equation (20), it would give extremely small velocities ($v_L \sim 10^{-16}$ cm s$^{-1}$). In this expression, $\tilde{E}$ is the microscopic induction field, $\sigma$ is the normal-state conductivity, and $H_{c2}$ the upper critical field of the superconductor. It assumes, in a laboratory superconductor, relaxation of the vortex-core electron distribution function to isotropy in the frame of the ion lattice. But no equivalent interaction exists in a proton superconductor and the theory is therefore inapplicable (Jones 1987, 1991: Harrison 1991).

Muslimov & Tsygan (1985) considered $v_L$ to be determined by the steady-state equilibrium between a buoyancy force (the derivative of the isotropic part of equation 18) and a viscous force derived from magnetic scattering of electrons by an isolated moving vortex (see also Harvey, Ruderman & Shaham 1986). These calculations gave $v_L \sim 10^{-10} - 10^{-8}$ cm s$^{-1}$ but were subject to the objection that the viscous force used was not correct for the problem concerned, which was the motion of the whole triangular lattice, with fairly small intervortex spacing $d = 4.9\times10^{-10}B_{12}^{-1/2}$ cm, through the electron gas (Jones 1987, 1991; Harrison 1991). These authors considered the coupling of the lattice with the electrons and recognized that, given the limited interactions occurring in a superfluid system well below the critical temperature, the electron distribution function must relax to isotropy in the rest frame of the lattice. Jones (1987) ne-

glected the supercurrent screening condition (equation 15) and is therefore seriously in error, as is Jones (1991) who obtained a result equivalent to equation (10) but then incorrectly identified the quantity $\mathbf{J}^e \cdot \delta\boldsymbol{\mathcal{E}}$ as the rate of dissipation. The final conclusions of both these papers must therefore be disregarded. Harrison (1991) considered the gravity-induced electric field (the term $\boldsymbol{\mathcal{E}}_0$ of equation 3) which must be present in both normal and superconducting systems. His treatment of the electron-lattice interaction is based on assumptions similar to those of Jones (1991). It includes a Lorentz force on the lattice, but differs from Jones (1987, 1991) and the present paper in not considering the proton superfluid Magnus force produced by the difference $\mathbf{v}_{p0} - \mathbf{v}_L$. This is an effect arising from classical hydrodynamics (see Nozières and Vinen 1966, Jones 1991) and there would appear to be no doubt of its existence. Harrison also observed that the buoyancy force must be included in the equation for hydrodynamic equilibrium, as in equation (19). His final conclusion is that $v_L \ll 10^{-16}$ cm s$^{-1}$.

The paper by Goldreich & Reisenegger (1992) is concerned primarily with matter whose composition is defined as being chemically homogeneous (neutrons and charged components limited to electrons and protons), without superfluidity. It introduces to neutron star physics the concept of ambipolar diffusion (see also Haensel, Urpin & Yakovlev 1990). This is movement of the charged components and magnetic flux relative to the neutral part of the system. It would also be of relevance to the intermediate state in proton type I superconductivity if that were present in the neutron star interior. These authors observe that, in analyzing such motion, it is essential to divide particle flux vectors $N_i\mathbf{v}_i$ into solenoidal and irrotational components. Then solenoidal motion, for chemically homogeneous systems, is limited only by a viscous force. Its velocity is $v^s_{ambip} \propto \tau_{pn}$, where $\tau_{pn}$ is derived from $p - n$ scattering and is, in the normal system, the principal collision relaxation time. (The distinction is that solenoidal components do not change local number densities, whereas irrotational components produce chemical potential gradients and an imbalance $\delta\mu = \delta\mu_p + \delta\mu_e - \delta\mu_n$, which is the basis for the relationship between stability and stratification emphasized by these authors.)

The structure of the liquid core may be more complex than we have assumed in the present paper. Even if type II superconductivity were present in the outer region, protons in the inner core might be of type I. Type II flux quanta $\phi_0$, on entering the inner region, would merge to form an intermediate state of the type I superconductor in which magnetic flux is confined to filaments of normal proton Fermi liquid with macroscopic cross-sectional area. The relaxation time $\tau_{pn}$ derived for proton interaction with superfluid neutrons (well below the critical temperature) is extremely long so it might be thought that there would be rapid ambipolar diffusion of these filaments under stresses generated by flux movement in the outer (type II) region. But this is not necessarily so owing to the presence, in many equations of state, of muons at inner-region matter densities. The matter is then chemically inhomogeneous in the sense considered by Goldreich & Reisenegger. Radial motion would produce a large chemical potential difference $\delta\mu_e - \delta\mu_\mu$ because the muons are non-relativistic. The consequence, following Goldreich & Reisenegger, is that even solenoidal particle fluxes are strongly inhibited by the stratification and stability condition so that flux movement out of the type I region would be slow, depending on weak-interaction transitions to remove the imbalance. The rates for these are strongly suppressed by the proton (and possibly also the neutron) energy gap; also by the requirement for energy-momentum conservation in direct $\mu \rightleftharpoons e$ transitions on the Fermi surfaces with neutrino-pair creation.

We emphasize that the flux velocity $\mathbf{v}_L$ given by equation (22) can be of an order of magnitude different from the individual drift velocities $\mathbf{v}_{e,\mu,p0}$. In this respect, the motion differs from the ambipolar diffusion described by Goldreich & Reisenegger. On the other hand, there is a similarity between equation (22) and ambipolar diffusion which is worth noting in relation to the proton-vortex drift velocity obtained by Ruderman, Zhu & Chen (1998; equations 10 and 14 of that paper) which is inversely proportional to an effective conductivity $\sigma$. It is defined by dissipation in an induction field rather than by a viscous force. The explanation for this appears to be that, in considering proton vortex motion, these authors have not included both the Magnus force and the screening condition given by equation (15) of the present paper. This appears to be the origin of the disagreement with our drift velocity $\mathbf{v}_L$. Also, the effective conductivity of Ruderman, Zhu & Chen is determined principally by the component of the magnetic force on an isolated moving proton vortex that is antiparallel with its velocity, and for that reason is referred to as the magnetic viscous or drag force. The view of the present paper (see also Jones 1991) is that this scattering is the process that produces the polygonal electron orbits and, of course, is the source of their irregularity. The discussion which immediately follows equation (3) expresses our view that this process, represented in equations (1) and (3) by the terms containing $\tilde{\mathbf{B}} - \mathbf{B}$, does not contribute to the Boltzmann collision integral.

The work of this paper has been limited to density regions containing only electrons, muons, protons and neutrons. Most equations of state have hyperon thresholds that are likely to be exceeded in a typical $1.4M_\odot$ neutron star. Thus a complete discussion of flux transport would need to consider not merely the possibility of type I proton superconductivity in the higher-density regions, but the physics of systems with two or more superconducting baryonic components.

## REFERENCES

Bardeen J., Stephen M. J., 1965, Phys. Rev. A, 140, 1197
Baym G., Pethick C. J., 1979, Ann. Rev. Astr. Astrophys., 17, 415
Baym G., Pethick C. J., Pines D., 1969, Nature, 224, 673
Bhattacharya D., Srinivasan G., 1995, in Lewin W. H. G., van Paradijs J., van den Heuvel E. P. J., eds., X-Ray Binaries. Cambridge University Press, p. 495
Buckley K. B. W., Metlitski M. A., Zhitnitsky A. R., 2004, Phys. Rev. Lett., 92, 151102
Easson I., Pethick C. J., 1977, Phys. Rev. D, 16, 275
Fetter A. L., Hohenberg P. C., 1969, in Parks R. D., ed., Superconductivity. M. Dekker, New York, p. 817
Gold T., 1968, Nature, 218, 731
Goldreich P., Reisenegger A., 1992, ApJ, 395, 250

Haensel P., Urpin V. A., Yakovlev D. G., 1990, A&A 229, 133
Harrison E., 1991, MNRAS, 248, 419
Harvey J. A., Ruderman M. A., Shaham J., 1986, Phys. Rev. D, 33, 2084
Jones P. B., 1975, Astrophys. Space Sci., 33, 215
Jones P. B., 1987, MNRAS, 228, 513
Jones P. B., 1991, MNRAS, 253, 279
Jones P. B., 2004, MNRAS, 351, 956
Konenkov D., Geppert U., 2001, A&A, 372, 583
Muslimov A. G., Tsygan A. I., 1985, Sov. Astr, Lett., 11, 80
Nozières P., Vinen W. F., 1966, Phil. Mag., 14, 667
Pacini F., 1967, Nature, 216, 567
Pethick C. J., Ravenhall D. G., 1995, Ann. Rev. Nucl. Part. Sci., 45, 429
Pines D., Nozières P., 1966, Theory of Quantum Liquids, Vol. I. W. A. Benjamin, New York
Potekhin A. Y., 1999, A&A, 351, 787
Ruderman M., Zhu T., Chen K., 1998, ApJ, 492, 267
Sauls J. A., 1989, Timing Neutron Stars, NATO ASI, 262, eds. Ögelman, H. & van den Heuvel, E. P. J. Kluwer, Dordrecht, p. 457
Spitzer L., 1968, Diffuse Matter in Space. John Wiley, New York, p. 238
Urpin V., Geppert U., Konenkov D., 1997, MNRAS, 295, 907
Verbunt F., van den Heuvel E. P. J., 1995, in Lewin W. H. G., van Paradijs J., van den Heuvel E. P. J., eds., X-Ray Binaries. Cambridge University Press, p. 457

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